

# NEW WEIGHTED HARDY'S INEQUALITIES WITH APPLICATION TO NONEXISTENCE OF GLOBAL SOLUTIONS

BY DANIEL HAUER, ABDELAZIZ RHANDI

ABSTRACT. In this article, we prove that the following weighted Hardy inequality

$$(1) \quad \left(\frac{|d-p|}{p}\right)^p \int \frac{|u|^p}{|x|^p} d\mu \leq \int |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \operatorname{sgn}(d-p) \int |u|^p \frac{(x^t A x)^{p/2}}{|x|^p} d\mu$$

holds for all  $u$  in the weighted Sobolev space  $W_\mu^{1,p}$  with best constant, where  $A \in \mathbb{R}^{d \times d}$  is a positive-definite symmetric matrix,  $d \geq 1$ ,  $1 < p < +\infty$ , and  $\mu$  denotes a Borel measure on  $\mathbb{R}^d$  given by

$$(2) \quad d\mu = \rho(x) dx \quad \text{with density} \quad \rho(x) = c \cdot \exp\left(-\frac{1}{p}(x^t A x)^{p/2}\right), \quad (c > 0).$$

Here the integral is taken over  $]0, +\infty[$  if  $d = 1$  and over  $\mathbb{R}^d$  if  $d \geq 2$ . If  $p > d$ , then we can deduce from inequality (1), that there is a Poincaré inequality on  $W_\mu^{1,p}$ . The proof of inequality (1) is based on the method of vector fields firstly introduced by Mitidieri [14]. By the same method, we prove for the same weights and for  $1 < p < +\infty$ ,  $d \geq 1$ , weighted Caffarelli-Kohn-Nirenberg inequalities. As an application of inequality (1) we prove a nonexistence result for a p-Kolmogorov parabolic equation.

## 1. Introduction

In the cases, when  $A \equiv 0$ , and  $c = 1$ , the Borel measure  $\mu$  defined in (2) reduces to the Lebesgue measure on  $\mathbb{R}^d$  and so inequality (1) becomes the well-known Hardy inequality

$$(3) \quad \left(\frac{|d-p|}{p}\right)^p \int \frac{|u|^p}{|x|^p} dx \leq \int |\nabla u|^p dx \quad (u \in W^{1,p}).$$

Inequality (3) was first stated in dimension  $d = 1$  in [13] by Hardy in 1920. Various generalization of Hardy's inequality since have been found with application to various branches of mathematics, see for instance [15, p. 175f] by Mitrinović, Pečarić, and Fink. Among them, we want shortly summarize the relation between the Hardy inequality with its optimal constant and the existence and nonexistence theory of nonnegative solutions of parabolic equations containing a critical potential. In 1984 Baras and J. Goldstein proved in [2] the following important result.

**THEOREM A. 1. (Baras-Goldstein)** *Let  $\Omega = ]0, +\infty[$  if  $d = 1$  and  $\Omega = \mathbb{R}^d$  if  $d \geq 2$ .*

(i) *If  $\lambda \leq \left(\frac{d-2}{2}\right)^2$ , then for the linear parabolic equation*

$$(4) \quad \frac{\partial u}{\partial t} - \Delta u = \frac{\lambda}{|x|^2} u \quad x \in \Omega, \quad t > 0,$$

*admits a nonnegative nontrivial solution.*

(ii) *If  $\lambda > \left(\frac{d-2}{2}\right)^2$ , then equation (4) admits no nonnegative nontrivial solution.*

Obviously, the phenomenon of existence and nonexistence is caused by the singular potential  $\lambda|x|^{-2}$ , which is controlled by Hardy's inequality (3) together with its optimal constant. Improvements of inequality (3) with or without reminder term of inequality (3) have been found, see for

---

*Date:* July 20, 2012.

*2010 Mathematics Subject Classification.* Primary 35A01, 35B09, 35B25, 35D30, 35K67, 35K92.

*Key words and phrases.* Hardy's inequality, nonlinear Ornstein-Uhlenbeck operator,  $p$ -Laplace operator, singular perturbation, existence, nonexistence, weak solution.

instance [4, 1997] by Brezis and Vázquez, [8, 1998] by García Azorero and Peral Alonso, [17, 2000] by Vázquez and Zuazua, [1, 2006] by W. Arendt, G. R. Goldstein, and J. A. Goldstein, as well as [10, 2011] by G. R. Goldstein, J. A. Goldstein and Rhandi, and served them in their studies about existence, nonexistence, and the qualitative behavior of solutions. In this way, for instance, the authors in [10], have established first inequality (1) when  $d \geq 3$ ,  $p = 2$ ,  $A \in \mathbb{R}^{d \times d}$  is a positive-definite symmetric matrix, and  $c$  is such that  $\int \rho \, dx = 1$ . Then, they proved the following result:

**THEOREM A. 2. (Goldstein-Goldstein-Rhandi)**

- (i) If  $\lambda \leq \left(\frac{d-2}{2}\right)^2$ , then for every nonnegative initial value  $u_0 \in L^2_\mu$ , equation
- $$(5) \quad \frac{\partial u}{\partial t} - \Delta u = \langle Ax, \nabla u \rangle + \frac{\lambda u}{|x|^2} \quad x \in \mathbb{R}^d, \quad t > 0$$
- admits a nonnegative global solution with exponential growth.
- (ii) If  $\lambda > \left(\frac{d-2}{2}\right)^2$ , then for every nonnegative initial value  $u_0 \in L^2_\mu \setminus \{0\}$  there is no nonnegative global solution of equation (5) with exponential growth.

In order to prove the nonexistence of global solutions of equation (5), the authors in [10] employ an approach in [5] due to Cabré and Martel. Comparing Theorem 2 with Theorem 1, one sees that the unbounded drift term  $Ax$  appearing in the symmetric Ornstein-Uhlenbeck operator  $L = -\Delta + \langle Ax, \nabla \cdot \rangle$ , has a strong influence on the qualitative behavior on the corresponding solutions of equation (5).

In analogs with the two theorems above, we first state and prove in Section 2 of this article the weighted Hardy inequality (1). If  $p > d$ , then we deduce from inequality (1) a weighted Poincaré inequality. The proof of inequality (1) is here based on the so-called method of vector field first introduced by Mitidieri in [14]. Based on the same method, we state weighted Caffarelli-Kohn-Nirenberg inequalities in Section 3. In the last section, we apply our Hardy inequality and use the optimality of the constant  $\left(\frac{|d-p|}{p}\right)^p$  to prove an existence and nonexistence result in dimension  $d = 1$  of equation (5), where we replace the symmetric Ornstein-Uhlenbeck operator  $L$  in (5) by the nonlinear  $p$ -Kolmogorov operator

$$K_p u = -\Delta_p + \langle |\nabla \cdot|^{p-2} \nabla \cdot, \frac{\nabla \rho}{\rho} \rangle.$$

The  $p$ -Kolmogorov operator was first introduced by G. R. Goldstein, J. A. Goldstein and Rhandi in [9]. We will use an idea of Cabré and Martel in [5].

Throughout this article, we employ the following notation: For  $1 \leq p \leq +\infty$  and if  $\mu$  is the Borel measure defined by (2), then we denote by  $L^p_\mu := L^p(d\mu)$  the weighted Lebesgue space on either  $]0, +\infty[$  if  $d = 1$  or on  $\mathbb{R}^d$  if  $d \geq 2$ , and by  $L^p_{\mu,loc}$  the set of locally  $L^p_\mu$ -integrable function on either  $]0, +\infty[$  if  $d = 1$  or on  $\mathbb{R}^d$  if  $d \geq 2$ . Further, we denote by  $W^{1,p}_\mu$  the first weighted Sobolev space, i.e., the set of all  $u \in L^p_\mu$  having the distributional partial derivative  $\partial_{x_1} u, \dots, \partial_{x_d} u \in L^p_\mu$ . Similarly, an element  $u \in W^{1,p}_{\mu,loc}$  if and only if  $u \in L^p_{\mu,loc}$  and  $\partial_{x_1} u, \dots, \partial_{x_d} u \in L^p_{\mu,loc}$ . In this context, it is worth mentioning, that for the measure  $\mu$  defined by (2), the set  $C_c^\infty$  of infinitely differentiable function having compact support in either  $]0, +\infty[$  if  $d = 1$  or on  $\mathbb{R}^d$  if  $d \geq 2$  lies dense in  $W^{1,p}_\mu$ , see for instance [16] by Tölle.

## 2. Main results

In this section, we study for  $1 < p < +\infty$  the following weighted Hardy inequality with optimal constant.

**THEOREM 2.1.** *Let  $A \in \mathbb{R}^{d \times d}$  be a symmetric positive semi-definite matrix. Then, for all  $u \in W_\mu^{1,p}$ ,*

$$(1) \quad \left(\frac{|d-p|}{p}\right)^p \int \frac{|u|^p}{|x|^p} d\mu \leq \int |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \operatorname{sgn}(d-p) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^p} d\mu.$$

Moreover, if either  $A \equiv 0$  or  $A$  is positive definite, then the constant  $C(d, p) = \left(\frac{|d-p|}{p}\right)^p$  is optimal (including the case  $p = d$  if  $d \geq 2$ ). Here the integral is taken over  $]0, +\infty[$  if  $d = 1$  and over  $\mathbb{R}^d$  if  $d \geq 2$ .

**REMARK 2.1.** Since for any symmetric positive semi-definite matrix  $A \in \mathbb{R}^{d \times d}$ , we have that  $(x^t A x) \leq |x|^2 |A|$  for all  $x \in \mathbb{R}^d$ , one easily sees that the second term on the right hand-side in (1) satisfies

$$\operatorname{sgn}(d-p) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^p} d\mu \leq |A|^{\frac{p}{2}} \int |u|^p d\mu \quad \text{for all } u \in L_\mu^p.$$

Thus Hardy's inequality (1) implies that  $W_\mu^{1,p} \hookrightarrow L^p(\frac{C(d,p)}{|x|^p} d\mu)$  by a continuous injection, provided  $p \neq d$ . Furthermore, we can deduce from inequality (1) the following Poincaré inequality.

**COROLLARY 2.2.** *If the matrix  $A$  is positive-definite, then for  $p > d$ ,*

$$(6) \quad \left(\frac{p-d}{p}\right)^{p-1} \lambda^{p/2}(A) \int |u|^p d\mu \leq \int |\nabla u|^p d\mu \quad \text{for all } u \in W_\mu^{1,p},$$

where  $\lambda(A) > 0$  denotes the lowest eigenvalue of  $A$ .

**PROOF OF COROLLARY 2.2.** If the symmetric matrix  $A \in \mathbb{R}^d$  is positive definite, then the lowest eigenvalue  $\lambda(A)$  of  $A$  is strictly positive and  $x^t A x \geq \lambda(A) |x|^2$  for all  $x \in \mathbb{R}^d$ , whence we can deduce inequality (6) from inequality (1) provided  $p > d$ .  $\square$

**PROOF OF THEOREM 2.1.** It is not hard to see that in the case  $p = d$  for  $d \geq 2$ , inequality (1) holds true with optimal constant  $C(d, d) = 0$ . It is left to show that inequality (1) holds true when  $p \neq d$  and  $d \geq 1$ . To do so, we follow an approach introduced by Mitidieri in [14]. We take  $\varepsilon > 0$ ,  $\lambda \geq 0$ , which will be chosen later, and set

$$F(x) = \lambda \operatorname{sgn}(d-p) \frac{x}{|x|^p + \varepsilon} \rho(x)$$

for every  $x \in \mathbb{R}^d$  if  $d \geq 2$  and for all  $x \in ]0, +\infty[$  if  $d \geq 1$ , where  $\rho$  is defined in (2). Since for every  $i = 1, \dots, d$ ,

$$\frac{\partial}{\partial x_i} \frac{x_i}{|x|^p + \varepsilon} = \frac{1}{|x|^p + \varepsilon} - \frac{p x_i^2 |x|^{p-2}}{(|x|^p + \varepsilon)^2}$$

and since for  $f(x) := x^t A x$ , ( $x \in \mathbb{R}^d$ ),  $f'(x) = x^t (A + A^t) = 2x^t A$ , we have that

$$\frac{\partial F_i}{\partial x_i} = \lambda \operatorname{sgn}(d-p) \left[ \frac{1}{|x|^p + \varepsilon} - \frac{p x_i^2 |x|^{p-2}}{(|x|^p + \varepsilon)^2} - \frac{x_i (x^t A)_i}{|x|^p + \varepsilon} (x^t A x)^{\frac{p}{2}-1} \right] \rho(x)$$

and so

$$\operatorname{div}(F(x)) = \lambda \operatorname{sgn}(d-p) \left[ \frac{d}{|x|^p + \varepsilon} - \frac{p |x|^p}{(|x|^p + \varepsilon)^2} - \frac{(x^t A x)^{\frac{p}{2}}}{|x|^p + \varepsilon} \right] \rho(x).$$

Since the set of infinitely differentiable functions with compact support lies dense in  $W_\mu^{1,p}$ , it is sufficient to show that inequality (1) holds for  $u \in C_c^\infty$ . Fix  $u \in C_c^\infty$ . Then, by an integration by parts and Young's inequality,

$$\begin{aligned} \int |u|^p \lambda \operatorname{sgn}(d-p) \left[ \frac{d}{|x|^{p+\varepsilon}} - \frac{p|x|^p}{(|x|^{p+\varepsilon})^2} - \frac{(x^t A x)^{\frac{p}{2}}}{|x|^{p+\varepsilon}} \right] d\mu &= \int |u|^p \operatorname{div}(F) dx \\ &= (-p) \int |u|^{p-1} \operatorname{sgn}(u) \langle \nabla u, F \rangle dx \\ &\leq \int |\nabla u|^p d\mu + (p-1) \lambda^{p'} \int \frac{|u|^p}{|x|^p} d\mu. \end{aligned}$$

And hence

$$\begin{aligned} \int |u|^p \lambda \operatorname{sgn}(d-p) \left[ \frac{d}{|x|^{p+\varepsilon}} - \frac{p|x|^p}{(|x|^{p+\varepsilon})^2} \right] d\mu - (p-1) \lambda^{p'} \int \frac{|u|^p}{|x|^p} d\mu \\ \leq \int |\nabla u|^p d\mu + \lambda \operatorname{sgn}(d-p) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^{p+\varepsilon}} d\mu. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0+$ , applying Lebesgue's dominated convergence theorem and Fatou's lemma, we obtain that

$$\int \frac{|u|^p}{|x|^p} [\lambda |d-p| - (p-1) \lambda^{p'}] d\mu \leq \int |\nabla u|^p d\mu + \lambda \operatorname{sgn}(d-p) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^p} d\mu.$$

Now, we choose  $\lambda = \left( \frac{|d-p|}{p} \right)^{p-1}$  in the last inequality, which is, in fact, the maximum of the function  $\lambda \mapsto [\lambda |d-p| - (p-1) \lambda^{p'}]$  on the half line  $[0, +\infty[$  and achieve to inequality (1).

Next, we show the optimality of the constant  $\left( \frac{|d-p|}{p} \right)^p$  when  $d \neq p$  and when the matrix  $A$  is positive definite. To do so, let  $\lambda > \left( \frac{|d-p|}{p} \right)^p$ , and for  $\gamma$  such that

$$(7) \quad 1 - \frac{d}{p} < \gamma < 0 \quad \text{if } p < d \quad \text{and} \quad 1 - \frac{d}{p} < \gamma < 1 \quad \text{if } p > d,$$

set  $\varphi(x) = |x|^\gamma$ . Then,  $\nabla \varphi(x) = \gamma |x|^{\gamma-2} x$  for all  $x \neq 0$ , and since  $x^t A x \leq |A| |x|^2$  for all  $x \in \mathbb{R}^d$ , we have that

$$\begin{aligned} (8) \quad \int \left[ |\nabla \varphi|^p + \lambda^{1/p'} \operatorname{sgn}(d-p) |\varphi|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^p} - \frac{\lambda}{|x|^p} |\varphi|^p \right] d\mu \\ \leq [|\gamma|^p - \lambda] \int |x|^{p(\gamma-1)} d\mu + |A|^{p/2} \lambda^{\frac{1}{p'}} \int |x|^{\gamma p} d\mu. \end{aligned}$$

By the assumption, there are  $\alpha_1, \alpha_2 > 0$  such that

$$\alpha_1 |x|^2 \leq x^t A x \leq \alpha_2 |x|^2 \quad \text{for all } x \in \mathbb{R}^d.$$

Hence for every  $\beta \in \mathbb{R}$ ,

$$(9) \quad \int |x|^{p\beta} d\mu \leq c \int_0^{+\infty} |x|^{p\beta} e^{-\frac{\alpha_1^{p/2} |x|^p}{p}} dx \quad \text{and} \quad \int |x|^{p\beta} d\mu \geq c \int_0^{+\infty} |x|^{p\beta} e^{-\frac{\alpha_2^{p/2} |x|^p}{p}} dx.$$

For every  $i = 1, 2$ , and every  $\beta \in \mathbb{R}$ ,

$$\begin{aligned} (10) \quad \int |x|^{p\beta} e^{-\frac{\alpha_i^{p/2} |x|^p}{p}} dx &= \sigma(S_{d-1}) \int_0^{+\infty} r^{p\beta} e^{-\frac{\alpha_i^{p/2} r^p}{p}} r^{d-1} dr \\ &= \sigma(S_{d-1}) p^{\frac{p\beta+d-p}{p}} \alpha_i^{-\frac{p\beta+d}{2}} \int_0^{+\infty} t^{\beta+\frac{d}{p}-1} e^{-t} dt, \end{aligned}$$

where  $\sigma(S_{d-1})$  denotes the total surface measure of the unite sphere  $S_{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$  with respect to the surface measure  $\sigma$  on  $S_{d-1}$ . We note that

$$\int_0^{+\infty} t^{\beta + \frac{d}{p} - 1} e^{-t} dt = \Gamma(\beta + \frac{d}{p}) \quad \text{is finite for every } \beta > -\frac{d}{p}$$

and in particular for  $\beta = \gamma$  or  $\beta = \gamma - 1$  when we choose  $\gamma$  as in (7). Thus,  $\varphi(x) = |x|^\gamma$  belongs to  $W_\mu^{1,p}$ , and in view of (8)-(10), we have that

$$\begin{aligned} & \frac{\int \left[ |\nabla \varphi|^p + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} - \frac{\lambda}{|x|^p} |\varphi|^p \right] d\mu}{\int |\varphi|^p d\mu} \\ & \leq \frac{[|\gamma|^p - \lambda] \int |x|^{p(\gamma-1)} d\mu + |A|^{p/2} \lambda^{\frac{1}{p'}} \int |x|^\gamma d\mu}{\int |x|^{p\gamma} d\mu} \\ & \leq \frac{[|\gamma|^p - \lambda] p^{\frac{p(\gamma-1)+d-p}{p}} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} \Gamma((\gamma-1) + \frac{d}{p}) + |A|^{p/2} \lambda^{\frac{1}{p'}} p^{\frac{p\gamma+d-p}{p}} \alpha_1^{-\frac{p\gamma+d}{2}} \Gamma(\gamma + \frac{d}{p})}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} \Gamma(\gamma + \frac{d}{p})} \\ & = \frac{[|\gamma|^p - \lambda] p^{\frac{p(\gamma-1)+d-p}{p}} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} + |A|^{p/2} \lambda^{\frac{1}{p'}} p^{\frac{p\gamma+d-p}{p}} \alpha_1^{-\frac{p\gamma+d}{2}} [(\gamma-1) + \frac{d}{p}]}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} [(\gamma-1) + \frac{d}{p}]} \end{aligned}$$

Since,

$$\begin{aligned} \lim_{\gamma \rightarrow (1-\frac{d}{p})+} p^{\frac{p(\gamma-1)+d-p}{p}} &= p^{-1}, & \lim_{\gamma \rightarrow (1-\frac{d}{p})+} p^{\frac{p\gamma+d-p}{p}} &= 1, & \lim_{\gamma \rightarrow (1-\frac{d}{p})+} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} &= 1, \\ \lim_{\gamma \rightarrow (1-\frac{d}{p})+} \alpha_i^{-\frac{p\gamma+d}{2}} &= \alpha_i^{-\frac{p}{2}} \quad (i = 1, 2), & \lim_{\gamma \rightarrow (1-\frac{d}{p})+} [(\gamma-1) + \frac{d}{p}] &= 0, \end{aligned}$$

and since

$$\lim_{\gamma \rightarrow (1-\frac{d}{p})+} (|\gamma|^p - \lambda) = \left(\frac{|d-p|}{p}\right)^p - \lambda < 0,$$

we have that

$$\lim_{\gamma \rightarrow (1-\frac{d}{p})+} \frac{[|\gamma|^p - \lambda] p^{\frac{p(\gamma-1)+d-p}{p}} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} + |A|^{p/2} \lambda^{\frac{1}{p'}} p^{\frac{p\gamma+d-p}{p}} \alpha_1^{-\frac{p\gamma+d}{2}} [(\gamma-1) + \frac{d}{p}]}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} [(\gamma-1) + \frac{d}{p}]} = -\infty.$$

Thus,

$$\inf_{\varphi \in W_\mu^{1,p} : \|\varphi\|_{L_\mu^p} > 0} \frac{\int |\nabla \varphi|^p d\mu + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) \int |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu - \int \frac{\lambda}{|x|^p} |\varphi|^p d\mu}{\int |\varphi|^p d\mu} = -\infty$$

and hence, for every  $M > 0$ , there is a  $\varphi \in W_\mu^{1,p}$  with  $\|\varphi\|_{L_\mu^p} > 0$  satisfying

$$\int |\nabla \varphi|^p d\mu + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) \int |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu - \int \frac{\lambda}{|x|^p} |\varphi|^p d\mu < (-M) \int |\varphi|^p d\mu < 0.$$

This shows that the constant  $(\frac{|d-p|}{p})^p$  in inequality (1) is optimal.  $\square$

### 3. Two weighted Caffarelli-Kohn-Nirenberg inequalities

We follow again the Ansatz as outlined in [14] and we prove for  $1 < p < +\infty$  the following two weighted inequalities with optimal constant. Both inequalities reduce in the case  $p = 1$ ,  $c = 1$ ,  $A \equiv 0$  to the famous Caffarelli-Kohn-Nirenberg inequalities [6]. Here and as above, the integral is taken over  $]0, +\infty[$  if  $d = 1$  and over  $\mathbb{R}^d$  if  $d \geq 2$ . We denote by  $C_{c,0}^\infty$  either the set  $C_c^\infty(]0, +\infty[)$  if  $d = 1$  or  $C_c^\infty(\mathbb{R}^d \setminus \{0\})$  if  $d \geq 2$ .

**THEOREM 3.1.** *Let  $A \in \mathbb{R}^{d \times d}$  be a symmetric positive semi-definite matrix, and let  $a \in \mathbb{R}$ . Then*

$$\left(\frac{|d-p(a+1)|}{p}\right)^p \int \frac{|u|^p}{|x|^{p(a+1)}} d\mu \leq \int \frac{|\nabla u|^p}{|x|^{pa}} d\mu + \left(\frac{|d-p(a+1)|}{p}\right)^{p-1} \operatorname{sgn}(d-p(a+1)) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^{p(a+1)}} d\mu$$

for all  $u \in C_{c,0}^\infty$ . Moreover, if either  $A \equiv 0$  or  $A$  is positive definite, then the constant  $\left(\frac{|d-p(a+1)|}{p}\right)^p$  is optimal (including the case  $p(a+1) = d$ ).

**PROOF OF THEOREM 3.1.** We take  $\lambda \geq 0$ , set  $F(x) = \lambda \operatorname{sgn}(d-p(a+1)) \frac{x}{|x|^{p(a+1)}} \rho(x)$ , and proceed analogously as in the proof of Theorem 2.1.  $\square$

**THEOREM 3.2.** *Let  $A \in \mathbb{R}^{d \times d}$  be a symmetric positive semi-definite matrix, and let  $\beta \in \mathbb{R}$ . Then*

$$\left(\frac{|d-(p+\beta)|}{p}\right)^p \int \frac{|u|^p}{|x|^{p+\beta}} d\mu \leq \int \frac{|\nabla u|^p}{|x|^\beta} d\mu + \left(\frac{|d-(p+\beta)|}{p}\right)^{p-1} \operatorname{sgn}(d-(p+\beta)) \int |u|^p \frac{(x^t A x)^{\frac{p}{2}}}{|x|^{p+\beta}} d\mu$$

for all  $u \in C_{c,0}^\infty$ . Moreover, if either  $A \equiv 0$  or  $A$  is positive definite, then the constant  $\left(\frac{|d-p(a+1)|}{p}\right)^p$  is optimal (including the case  $p(a+1) = d$ ).

**PROOF OF THEOREM 3.2.** For  $\lambda \geq 0$ , we set  $F(x) = \lambda \operatorname{sgn}(d-(p+\beta)) \frac{x}{|x|^{p+\beta}} \rho(x)$ , and proceed analogously as in the proof of Theorem 2.1.  $\square$

### 4. Application: A nonexistence result

In this section, we prove existence and nonexistence of nonnegative solutions to the p-Kolmogorov parabolic problem (13). We define solutions in the following sense. Similar definitions can be found for example in [12] or [7].

**DEFINITION 4.1.** Let  $u_0 \in L_\mu^2(]0, +\infty[)$ ,  $f \in L^2(0, T; L_\mu^2(]0, +\infty[))$ , and let  $\lambda > 0$ . We call a function

$$u \in C([0, T]; L_\mu^2(]0, +\infty[)) \cap L^p(0, T; W_{\mu, \text{loc}}^{1,p}(]0, +\infty[))$$

a weak solution locally off of zero of equation

$$(11) \quad \partial_t u - \partial_x \{ |\partial_x u|^{p-2} \partial_x u \} = |\partial_x u|^{p-2} \partial_x u \frac{\partial_x \rho}{\rho} + \frac{\lambda}{x^p} |u|^{p-2} u + f \quad \text{on } ]0, +\infty[ \times ]0, T[,$$

if for all  $\mathcal{K} \Subset ]0, +\infty[$ ,  $t_1, t_2 \in [0, T] : t_1 \leq t_2$ , and all  $\varphi \in W^{1,2}(t_1, t_2; L_\mu^2(\mathcal{K})) \cap L^p(t_1, t_2; W_{\mu,0}^{1,p}(\mathcal{K}))$ ,

$$(12) \quad \begin{aligned} & (u, \varphi)_{L_\mu^2(\mathcal{K})} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} \left\{ -u \partial_t \varphi + |\partial_x u|^{p-2} \partial_x u \partial_x \varphi - \frac{\lambda}{x^p} |u|^{p-2} u \varphi \right\} d\mu dt \\ & = \int_{t_1}^{t_2} \int_{\mathcal{K}} f \varphi d\mu dt. \end{aligned}$$

If such a function  $u$  is nonnegative a.e. on  $]0, +\infty[ \times ]0, T[$ , then we call  $u$  a nonnegative weak solution locally off of zero of equation (11). We call a function  $u$  a *weak solution locally off of zero of initial value problem*

$$(13) \quad \begin{cases} \partial_t u - \partial_x \{ |\partial_x u|^{p-2} \partial_x u \} &= |\partial_x u|^{p-2} \partial_x u \frac{\partial_x \rho}{\rho} + \frac{\lambda}{x^p} |u|^{p-2} u + f & \text{on } ]0, +\infty[ \times ]0, T[, \\ u(0) &= u_0 & \text{on } ]0, +\infty[, \end{cases}$$

if  $u$  is a weak solution locally off of zero of equation (11) and satisfies  $u(0) = u_0$  in  $L_\mu^2([0, +\infty[)$ . Here we stress that since  $u \in C([0, +\infty[; L_\mu^2([0, +\infty[))$ , the initial condition  $u(0) = u_0$  in  $L_\mu^2([0, +\infty[)$  has a meaning. We call a weak solutions  $u$  locally off of zero of initial value problem (13) *global* if

$$u \in C([0, +\infty[; L_\mu^2([0, +\infty[)) \cap L_{loc}^p(0, +\infty; W_{\mu, loc}^{1,p}([0, +\infty[)).$$

**THEOREM 4.1.** *Then the following assertions are true:*

- (i) *If  $0 \leq \lambda \leq C(1, p)$  and if  $1 < p < +\infty$ , then for every nonnegative  $u_0 \in L_\mu^2([0, +\infty[)$  and for every nonnegative  $f \in L^2(0, +\infty; L_\mu^2([0, +\infty[))$ , there is at least one global nonnegative weak solution  $u$  off of zero of initial value problem (13) satisfying*

$$(14) \quad \|u(t)\|_{L_\mu^2([0, +\infty[)} \leq \|u_0\|_{L_\mu^2} + \int_0^t \|f(s)\|_{L_\mu^2([0, +\infty[)} ds \quad \text{for all } t \geq 0.$$

- (ii) *If  $\lambda > C(1, p)$ , if  $1 < p < 2$  and if  $f \equiv 0$ , then for any nonnegative  $u_0 \in L_\mu^2([0, +\infty[) \setminus \{0\}$ , there is no global nonnegative weak solution  $u$  off of zero of initial value problem (13), which is bounded with values in  $L_\mu^2([0, +\infty[)$ .*

In order to prove the second assertion of Theorem 4.1, we need to introduce the Steklov average of a function  $v \in L^1([a, b[ \times ]\tau_1, \tau_2[)$  defined on  $]a, b[ \times ]\tau_1, \tau_2[ \subseteq \mathbb{R}^2$ : if  $0 < \varepsilon < \tau_2 - \tau_1$  and if  $0 < h < \varepsilon$ , then the Steklov average of  $v$  is given by  $v_h(x, t) := \frac{1}{h} \int_t^{t+h} v(x, s) ds$  for all  $t \in ]\tau_1, \tau_2 - \varepsilon[$ , a.e.  $x \in ]a, b[$ . Moreover, For  $q, r \geq 1$  we denote by  $L^{q,r}([a, b[ \times ]\tau_1, \tau_2[)$  the parabolic Lebesgue space  $L^r(\tau_1, \tau_2; L^q([a, b[))$  equipped with the norm

$$\|u\|_{L^{q,r}([a, b[ \times ]\tau_1, \tau_2[)} := \left( \int_{\tau_1}^{\tau_2} \left( \int_a^b |u(x, t)|^q dx \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}} \quad \text{for all } u \in L^{q,r}([a, b[ \times ]\tau_1, \tau_2[).$$

Furthermore, we need the following two well-known lemmas. For a proof, we refer the interested reader, for instance, to the book [7] of DiBenedetto.

**LEMMA 4.2.** *Let  $]a, b[, ]\tau_1, \tau_2[ \subseteq \mathbb{R}$  be two open intervals. Then the following assertions are valid.*

- (i) *If  $v \in L^{q,r}([a, b[ \times ]\tau_1, \tau_2[)$ , then for every  $0 < \varepsilon < \tau_2 - \tau_1$ ,*

$$v_h \rightarrow v \quad \text{in } L^{q,r}([a, b[ \times ]\tau_1, \tau_2 - \varepsilon[) \quad \text{as } h \rightarrow 0+.$$

- (ii) *If  $v \in C([ \tau_1, \tau_2 ]; L^q([a, b[))$ , then  $v_h(t)$  can be defined in  $t = \tau_1$  by  $v_h(x, \tau_1) = \frac{1}{h} \int_{\tau_1}^{\tau_1+h} v(x, s) ds$  ( $x \in ]a, b[$ ) for all  $0 < h < \tau_2 - \tau_1$ . Moreover, then for every  $0 < \varepsilon < \tau_2 - \tau_1$  and every  $t \in [\tau_1, \tau_2 - \varepsilon[$ ,*

$$v_h(t) \rightarrow v(t) \quad \text{in } L^q([a, b[) \quad \text{as } h \rightarrow 0+.$$

**LEMMA 4.3.** *If  $u$  is a weak solution locally off of zero of equation (11) and if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz-continuous, then for every  $\phi \in W_\mu^{1,p}([0, +\infty[) \cap L_\mu^2([0, +\infty[)$  having compact support contained in*

$]0, +\infty[$ , and every  $t_1, t_2 \in [0, T] : t_1 < t_2$ ,

$$\begin{aligned}
 & \int_0^{+\infty} \left( \int_0^{u(t_2)} g(s) ds \right) \phi d\mu - \int_0^{+\infty} \left( \int_0^{u(t_1)} g(s) ds \right) \phi d\mu \\
 (15) \quad & + \int_{t_1}^{t_2} \int_0^{+\infty} |\partial_x u|^p g'(u) \phi d\mu dt + \int_{t_1}^{t_2} \int_0^{+\infty} |\partial_x u|^{p-2} \partial_x u g(u) \partial_x \phi d\mu dt \\
 & = \int_{t_1}^{t_2} \int_0^{+\infty} \left( \frac{\lambda}{x^p} |u|^{p-2} u + f \right) g(u) \phi d\mu dt.
 \end{aligned}$$

**PROOF.** By a standard approximation argument, one sees that it is sufficient to prove the claim of this lemma for test-functions  $\phi \in C_c^1([0, +\infty[)$ . Thus, we fix  $\phi \in C_c^1([0, +\infty[)$ , and for fixed  $0 < t < t+h < T$ , we take  $t_1 = t$ ,  $t_2 = t+h$ , and multiply equation (12) by  $h^{-1}$ . Then equation (12) becomes

$$\begin{aligned}
 & \int_0^{+\infty} h^{-1} (u(t+h) - u(t)) \phi d\mu + \int_t^{t+h} \int_0^{+\infty} h^{-1} (|\partial_x u|^{p-2} \partial_x u) \partial_x \phi d\mu dt \\
 & = \int_t^{t+h} \int_0^{+\infty} h^{-1} \left( \frac{\lambda}{x^p} |u|^{p-2} u + f \right) \phi d\mu dt.
 \end{aligned}$$

By Fubini's theorem, since  $\partial_t u_h(t) = h^{-1} (u(t+h) - u(t))$ , and by the definition of Steklov averages, the last equality can be rewritten as

$$\int_0^{+\infty} \partial_t u_h(t) \phi d\mu + \int_0^{+\infty} (|\partial_x u|^{p-2} \partial_x u)_h(t) \partial_x \phi d\mu = \int_0^{+\infty} \left( \frac{\lambda}{x^p} |u|^{p-2} u + f \right)_h(t) \phi d\mu.$$

By the hypothesis, for any  $t \in ]0, T[$ ,  $g(u_h(t)) \phi \in W_{\mu,0}^{1,p}(\mathcal{K})$  with distributional derivative

$$\partial_x (g(u_h(t)) \phi) = g'(u_h(t)) (\partial_x u)_h(t) \phi + g(u_h(t)) \partial_x \phi,$$

where  $\mathcal{K} \Subset ]0, +\infty[$  can be chosen as an open and bounded interval such that the support of  $\phi$   $\text{supp}(\phi) \subseteq \mathcal{K}$ . Thus, we can replace  $\phi$  by  $g(u_h(t)) \phi$  in the last equality. Then, integrating over  $]t_1, t_2[$  for any  $t_1, t_2 \in [0, T] : t_1 < t_2$  and applying Fubini's theorem gives

$$\begin{aligned}
 & \int_0^{+\infty} \left( \int_0^{u_h(t_2)} g(s) ds \right) \phi d\mu - \int_0^{+\infty} \left( \int_0^{u_h(t_1)} g(s) ds \right) \phi d\mu \\
 & + \int_{t_1}^{t_2} \int_0^{+\infty} \left\{ (|\partial_x u|^{p-2} \partial_x u)_h(t) g'(u_h(t)) (\partial_x u)_h(t) \phi + (|\partial_x u|^{p-2} \partial_x u)_h(t) g(u_h(t)) \partial_x \phi \right\} d\mu dt \\
 & = \int_{t_1}^{t_2} \int_0^{+\infty} \left( \frac{\lambda}{x^p} |u|^{p-2} u + f \right)_h(t) g(u_h(t)) \phi d\mu dt.
 \end{aligned}$$

Now, by sending  $h \rightarrow 0+$  in the last equality and using Lemma 4.2 leads to equality (15).  $\square$

**PROOF OF THEOREM 4.1.** If  $\lambda \leq C(1, p)$ , then by Theorem 3.17 in [11], for every nonnegative  $u_0 \in L_\mu^2([0, +\infty[)$  and for every nonnegative  $f \in L^2(0, +\infty; L_\mu^2([0, +\infty[))$ , there exists at least one global nonnegative weak solution  $u$  off of zero of initial value problem (13). Moreover, there is a sequence  $(u_m)_{m \geq 1}$  of strong solutions

$$u_m \in W_{loc}^{1,2}([0, +\infty[; L_\mu^2([0, +\infty[)) \cap L_{loc}^p(0, +\infty; W_\mu^{1,p}([0, +\infty[))$$

of the truncated problem

$$(1_m) \quad \partial_t u_m - \partial_x \{ |\partial_x u_m|^{p-2} \partial_x u_m \} = |\partial_x u_m|^{p-2} \partial_x u_m \frac{\partial_x \rho}{\rho} + \Phi_m u_m^{p-1} + f_m \quad \text{on } ]0, +\infty[ \times ]0, +\infty[,$$

$$(2_m) \quad u_m(0) = u_m^0 \quad \text{on } ]0, +\infty[,$$



for  $\Phi_m(x) := \min\{\frac{\lambda}{x^p}, m\}$ ,  $f_m(x, t) = \min\{f(x, t), m\}$ , and  $u_m^0(x) := \min\{u_0(x), m\}$ , each of them defined pointwise almost everywhere, such that for every  $T > 0$ ,

$$(16) \quad u_m \rightarrow u \quad \text{in } C([0, T]; L_\mu^2([0, +\infty[)) \text{ as } m \rightarrow +\infty.$$

If we first multiply equation (1<sub>m</sub>) by  $u_m$  with respect to the inner product on  $L_\mu^2([0, +\infty[)$ , and subsequently integrate from 0 to  $t$  for fixed  $t \geq 0$ , then we obtain by Hardy's inequality (1) and by the Cauchy-Schwarz inequality that

$$\frac{1}{2} \|u_m(t)\|_{L_\mu^2([0, +\infty[)}^2 \leq \frac{1}{2} \|u_m^0\|_{L_\mu^2([0, +\infty[)}^2 + \int_0^t \|f_m(s)\|_{L_\mu^2([0, +\infty[)} \|u_m(s)\|_{L_\mu^2([0, +\infty[)} ds.$$

By Lemma A.5 in the book [3] by Brezis, we can deduce from the last inequality that

$$\|u_m(t)\|_{L_\mu^2([0, +\infty[)} \leq \|u_0\|_{L_\mu^2([0, +\infty[)} + \int_0^t \|f(s)\|_{L_\mu^2([0, +\infty[)} ds \quad \text{for all } t \geq 0.$$

Since  $\|u_m^0\|_{L_\mu^2([0, +\infty[)} \leq \|u_0\|_{L_\mu^2([0, +\infty[)}$  and  $\|f_m\|_{L^2(0, T, L_\mu^2([0, +\infty[))} \leq \|f\|_{L^2(0, T, L_\mu^2([0, +\infty[))}$ , and by limit (16), we can send  $m \rightarrow +\infty$  in the last inequality and see that  $u$  satisfies inequality (14). Thus claim (i) of this theorem holds true.

Let  $1 < p < 2$  and  $f \equiv 0$ . We suppose that there is a global nonnegative weak solution  $u$  off of zero of equation (11), with nonnegative initial value  $u(0) = u_0 \in L_\mu^2([0, +\infty[) \setminus \{0\}$ , which is bounded with values in  $L_\mu^2([0, +\infty[)$ . Then, we shall reach a contradiction. To see this, we fix  $\varphi \in C_c^\infty([0, +\infty[)$  and for every integer  $k \geq 1$  and every  $s \in \mathbb{R}$ , let  $g_k(s) = (s + 1/k)^{1-p}$ . Then, by Lemma 4.3 for  $g = g_k$ ,  $\phi = |\varphi|^p$ ,  $t_1 = 0$ , and  $t_2 = t$  for any fixed  $t \geq 0$ ,

$$\begin{aligned} & \frac{1}{2-p} \int_0^{+\infty} (u(t) + 1/k)^{2-p} |\varphi|^p d\mu - \frac{1}{2-p} \int_0^{+\infty} (u_0 + 1/k)^{2-p} |\varphi|^p d\mu \\ & + (1-p) \int_0^t \int_0^{+\infty} |\partial_x u(s)|^p (u(s) + 1/k)^{-p} |\varphi|^p d\mu ds \\ & + p \int_0^t \int_0^{+\infty} |\partial_x u(s)|^{p-2} \partial_x u(s) (u(s) + 1/k)^{1-p} \partial_x \varphi |\varphi|^{p-2} \varphi d\mu ds \\ & = \int_0^t \int_0^{+\infty} \frac{\lambda}{x^p} u^{p-1}(s) (u(s) + 1/k)^{1-p} |\varphi|^p d\mu ds. \end{aligned}$$

We apply Young's inequality, and since  $(u_0 + 1/k)^{2-p} |\varphi|^p$  is nonnegative a.e. on  $]0, +\infty[$ , we obtain that

$$(17) \quad \begin{aligned} & \int_0^t \int_0^{+\infty} \frac{\lambda}{x^p} u^{p-1}(s) (u(s) + 1/k)^{1-p} |\varphi|^p d\mu ds \\ & \leq t \int_0^{+\infty} |\partial_x \varphi|^p d\mu + \frac{1}{2-p} \int_0^{+\infty} (u(t) + 1/k)^{2-p} |\varphi|^p d\mu. \end{aligned}$$

For almost every  $(x, s) \in ]0, +\infty[ \times ]0, t[$ ,

$$0 \leq \frac{\lambda}{|x|^p} \frac{u^{p-1}(x, s) |\varphi(x)|^p}{(u(x, s) + \frac{1}{k})^{p-1}} \nearrow \frac{\lambda}{|x|^p} |\varphi(x)|^p \quad \text{as } k \rightarrow +\infty,$$

and

$$(u(x, s) + \frac{1}{k})^{2-p} |\varphi(x)|^p \searrow u^{2-p}(x, s) |\varphi(x)|^p \quad \text{as } k \rightarrow +\infty.$$

Thus, by Beppo-Levi's convergence theorem, sending  $k \rightarrow +\infty$  in inequality (17) gives

$$t \int_0^{+\infty} \frac{\lambda}{x^p} |\varphi|^p d\mu - t \int_0^{+\infty} |\partial_x \varphi|^p d\mu \leq \frac{1}{2-p} \int_0^{+\infty} u^{2-p}(t) |\varphi|^p d\mu,$$

and by Hölder's inequality,

$$t \int_0^{+\infty} \frac{\lambda}{x^p} |\varphi|^p d\mu - t \int_0^{+\infty} |\partial_x \varphi|^p d\mu \leq \frac{1}{2-p} \leq \|\varphi\|_{L^2([0, +\infty[)}^p \|u(t)\|_{L_\mu^2([0, +\infty[)}^{2-p}.$$

We divide this inequality by  $t > 0$ . Since  $t \mapsto u(t)$  is bounded from  $]0, +\infty[$  to  $L^2_\mu(]0, +\infty[)$ , sending  $t + \infty$  yields to

$$\int_0^{+\infty} \frac{\lambda}{x^p} |\varphi|^p d\mu - \int_0^{+\infty} |\partial_x \varphi|^p d\mu \leq 0.$$

Since  $\varphi \in C_c^\infty(]0, +\infty[)$  has been arbitrary in this inequality and since  $C_c^\infty(]0, +\infty[)$  lies dense in  $W_\mu^{1,p}(]0, +\infty[)$ , we have thereby shown that

$$\inf_{\varphi \in W_\mu^{1,p}(]0, +\infty[): \|\varphi\|_{L_\mu^p(]0, +\infty[)} > 0} \frac{\int_0^{+\infty} |\partial_x \varphi|^p d\mu - \int_0^{+\infty} \frac{\lambda}{x^p} |\varphi|^p d\mu}{\int_0^{+\infty} |\varphi|^p d\mu} \geq 0,$$

but this obviously contradicts the optimality of the constant  $C(1, p)$ .  $\square$

## References

- [1] W. Arendt, G. R. Goldstein, and J. A. Goldstein, *Outgrowth of Hardy's inequality*, Contemp. Math. **412** (2006), 51–68.
- [2] P. Baras and J. A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 121–139.
- [3] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973, North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).
- [4] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.
- [5] X. Cabré and Y. Martel, *Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 11, 973–978.
- [6] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275.
- [7] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.
- [8] J. P. García Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations **144** (1998), no. 2, 441–476.
- [9] G. R. Goldstein, J. A. Goldstein, and A. Rhandi, *Kolmogorov equations perturbed by an inverse-square potential*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), no. 3, 623–630.
- [10] G. R. Goldstein, J. A. Goldstein, and A. Rhandi, *Weighted Hardy's inequality and the Kolmogorov equation perturbed by an inverse-square potential*, Applicable Analysis, Doi: 10.1080/00036811.2011.587809 (2011), 1–15.
- [11] J. A. Goldstein, D. Hauer, and A. Rhandi, *On the existence and regularity of nonlinear parabolic equation with a singular potential*, in preperation (2012), 1–33.
- [12] J. A. Goldstein and I. Kombe, *Nonlinear degenerate parabolic equations with singular lower-order term*, Adv. Differential Equations **8** (2003), no. 10, 1153–1192.
- [13] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317.
- [14] E. Mitidieri, *A simple approach to Hardy inequalities*, Mat. Zametki **67** (2000), no. 4, 563–572.
- [15] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Mathematics and its Applications (East European Series), vol. 53, Kluwer Academic Publishers Group, Dordrecht, 1991.
- [16] J. M. Tölle, *Uniqueness of weighted Sobolev spaces with weakly differentiable weights*, (2011).
- [17] J. L. Vázquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. **173** (2000), no. 1, 103–153.

INSTITUT FÜR ANGEWANDTE ANALYSIS, UNIVERSITÄT ULM, 89069 ULM, GERMANY.  
 UNIVERSITÉ DE LORRAINE (CAMPUS DE METZ) ET CNRS, LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS DE  
 METZ, UMR 7122, BÂT. A, ILE DU SAULCY, 57045 METZ CEDEX 1, FRANCE.  
*E-mail address:* daniel.hauer@uni-ulm.de, hauer@math.cnrs.fr.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI SALERNO, VIA PONTE DON MELILLO, 84084 FIS-  
 CIANO (SA), ITALY.  
*E-mail address:* arhandi@unisa.it.